RESEARCH NOTES

CM elliptic curves and p-adic valuations of their L-series at $s = 1$*

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Received January 11, 2002; revised February 25, 2002

Abstract For rational integers γ and λ, consider two families of elliptic curves $y^2 = x^3 - D_1^\gamma x$ and $y^2 = x^3 - 2^4 3^3 D_2^\lambda$ over fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ respectively. General formulae expressed by Weierstrass $\wp$-functions are given for special values of Hecke L-series attached to such elliptic curves. The uniform lower bounds of 2-adic and 3-adic valuations of these values of Hecke L-series as well as global criteria for reaching these bounds are obtained. Moreover, when $\gamma = 2$ and $\lambda = 2, 4$, further results of 2-adic and 3-adic valuations are obtained for the corresponding curves in more general case of D_1 and some restricted D_2 respectively. These results are consistent with the predictions of the conjecture of Birch and Swinnerton-Dyer, and greatly develop and generalize some results in recent literature for more special cases.

Keywords: **elliptic curve, L-series, complex multiplication, Birch and Swinnerton-Dyer conjecture.**

The problem of p-power ($p = 2, 3$) divisibility of the L-series at $s = 1$ of the two classical families of elliptic curves E_A: $y^2 = x^3 - Ax$ and E_B: $y^2 = x^3 - 2^4 3^3 B$, have been extensively studied for ages, related to several problems of number theory such as congruent numbers (see e.g. Refs. [1 ~ 7] and footnotes 1), 2)).

These two elliptic curves E ($= E_A$, E_B) have complex multiplication by $\sqrt{-1}$ and $\sqrt{-3}$ respectively. From complex multiplication theory (see e.g. Refs. [8,9]), we know that as they are defined over $\mathbb{Q}$, the rational number field, so none of them have multiplicative reduction at any finite place v. The Kodaira-Neron theorem asserts that the Tamagawa numbers $c_v = 1$, 2, 3, or 4 (see Ref. [8], Theorem 6.1). So the "Birch and Swinnerton-Dyer conjecture"[8] (or "B-SD conjecture") predicts that the value $L(E/\mathbb{Q}, 1)$ of their L-series $L(E/\mathbb{Q}, s)$ at $s = 1$, when divided by an appropriate period, should always be divisible by a certain power of 2 or 3, depending on the number of distinct prime factors of the conductor $N(E)$ of E. It is the very problem of divisibility that we study in this paper, but in more general cases that E_A and E_B are defined over the fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ respectively.

Here we study the two families E_A and E_B of elliptic curves over the fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ respectively in the general cases, i.e. one is $E_{D_1^\gamma}$: $y^2 = x^3 - D_1^\gamma x$ with rational integer $\gamma \not\equiv 0 \pmod 4$ (i.e. 4 does not divide γ) and $D_1 = \pi_1 \cdots \pi_n$, where $\pi_1, \cdots, \pi_n$ are distinct primes in $\mathbb{Z}[\sqrt{-1}]$ (in particular, when $\gamma = 2$, this turns out to be the case studied in Ref. [6]). And the other is $E_{D_2^\lambda}$: $y^2 = x^3 - 2^4 3^3 D_2^\lambda$ with rational integer $\lambda \equiv \pm 2 \pmod 6$ and $D_2 = \pi_1 \cdots \pi_n$, where $\pi_1, \cdots, \pi_n$ are distinct prime integers in $\mathbb{Q}(\sqrt{-3})$. We give respectively general formulae for values at $s = 1$ of the Hecke L-series attached to $E_{D_1^\gamma}$ and $E_{D_2^\lambda}$, uniform lower bounds for 2-adic and 3-adic valuations of the values, and global criteria for reaching these bounds. In particular, in the case of $\gamma = 2$, we obtain further results for $E_A = E_{D_1^2}$ as studied in Ref. [6], but in more general case of D_1 (i.e. in the case of Gaussian

* Supported by China Postdoctoral Science Foundation, the Center for Advanced Study of Tsinghua University and the Morningside Center of Mathematics of CAS

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1) Qiu, D. R. et al. L-series and their 2-adic valuations at $s = 1$ attached to CM elliptic curves. To appear in Acta Arithmetica.

2) Qiu, D. R. et al. Elliptic curves with CM by $\sqrt{-3}$ and 3-adic valuations of their L-series. To appear in Manuscripta Mathematica.

primitive odd integer D_1). Further results are also obtained for $E_B = E_{D_2^\lambda}$ with restricted D_2.

Our results develop the ones in Refs. [6,7] and footnote 1) about 2-adic valuations for values of Hecke L-series of E_A with $A = D_1^2$ (i.e. $\gamma = 2$) and D_1 (i.e. $\gamma = 1$) in Gaussian field respectively, and develop the results in Refs. [2,7] and footnote 2) on the previous page about 3-divisibility and 3-adic valuation for values of L-series and Hecke L-series of E_B with $B = D_2^2$ (i.e. $\lambda = 2$) over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-3})$ respectively. In particular, our results are consistent with the predictions of B-SD conjecture.

1 L-series attached to curves $y^2 = x^3 - D^\gamma x$

Throughout, we put $I = \sqrt{-1}$. First we study the family of elliptic curves E_{D^γ}: $y^2 = x^3 - D^\gamma x$ over the Gaussian field $K = \mathbb{Q}(\sqrt{-1})$.

Consider the elliptic curves

E_{D^γ}: $y^2 = x^3 - D^\gamma x$ with $D = \pi_1 \cdots \pi_n$,

where $\pi_k \equiv 1 \pmod 4$ are distinct (Gaussian) prime integers in $O_K = \mathbb{Z}[\sqrt{-1}]$ $(k = 1, \cdots, n)$, and $\gamma \not\equiv 0 \pmod 4$ is a rational integer in $\mathbb{Z}$ (when $\gamma \equiv 0 \pmod 4$, E_{D^γ} is $\mathbb{Q}(\sqrt{-1})$-isomorphic to E_1: $y^2 = x^3 - x$, which is the trivial case. See Remark 1.(iii) in the following. Let $S = \{\pi_1, \cdots, \pi_n\}$. For any subset T of the set $\{1, \cdots, n\}$, define $D_T = \prod_{k \in T} \pi_k$, $\hat{D}_T = D/D_T$, and put $D_\varnothing = 1$ when $T = \varnothing$ (empty set). Let $\psi_{D_T^\gamma}$ be the Hecke character (Grössencharacter) of K attached to elliptic curve $E_{D_T^\gamma}$: $y^2 = x^3 - D_T^\gamma x$, and $L_S(\bar{\psi}_{D_T^\gamma}, s)$ be the Hecke L-series of $\bar{\psi}_{D_T^\gamma}$ with the Euler factors omitted at all primes in S (here $\bar{\psi}_{D_T^\gamma}$ is the complex conjugate of $\psi_{D_T^\gamma}$). For the definition of such Hecke L-series attached to an elliptic curve, see Ref. [9]. We have the following general formula for the special value $L_S(\bar{\psi}_{D_T^\gamma}, 1)$ of the above L-series at $s = 1$ expressed as a finite sum of Weierstrass $\wp$-function $\wp(z)$.

Theorem 1. For any factor D_T of $D = \pi_1 \cdots \pi_n \in \mathbb{Z}[\sqrt{-1}]$ and any rational integer $\gamma \not\equiv 0 \pmod 4$ as above, let $\psi_{D_T^\gamma}$ be the Hecke character of $\mathbb{Q}(\sqrt{-1})$ attached to the elliptic curve $E_{D_T^\gamma}$: $y^2 = x^3 - D_T^\gamma x$. Then we have

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4^\gamma} L_S(\bar{\psi}_{D_T^\gamma}, 1) = \frac{I}{2}\sum_{c \in C}\left(\frac{c}{D_T}\right)_4^\gamma \frac{1}{\wp\left(\frac{c\omega}{D}\right) - I} + \frac{1}{4}\sum_{c \in C}\left(\frac{c}{D_T}\right)_4^\gamma,$$

where $\theta = 2 + 2I$, $(-)_4$ is the quartic residue symbol, C is any complete set of representatives of the relatively prime residue classes of O_K modulo D, $\omega O_K = L_\omega$ is the period lattice of the elliptic curve E_1: $y^2 = x^3 - x$,

$$\omega = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} = 2.6220575\cdots,$$

and $\wp(z)$ is the Weierstrass $\wp$-function associated to the lattice L_ω (i.e., $\wp(z)$ and its derivative $\wp'(z)$ satisfies the equation $\wp'(z)^2 = 4\wp(z)^3 - 4\wp(z)$).

Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ at p-adic valuation for any rational prime number p, $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ be the algebraic closures of $\mathbb{Q}$ and $\mathbb{Q}_p$ respectively; and let v_p be the normalized p-adic additive valuation of $\overline{\mathbb{Q}}_p$ (i.e. $v_p(p) = 1$). Fix an isomorphic embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Then $v_p(\alpha)$ is defined for any algebraic number α in $\overline{\mathbb{Q}}$. The value $v_p(\alpha)$ for $\alpha \in \overline{\mathbb{Q}}$ depends on the choice of the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, but this does not affect our discussion in this paper. We will discuss the two cases: $p = 2$ and 3.

For any Gaussian integers α, β which are relatively prime, denote $(\alpha/\beta)_4^2 = (\alpha/\beta)_2$, $[\alpha/\beta]_2 = (1 - (\alpha/\beta)_2)/2$, then $[\alpha\gamma/\beta]_2 = [\alpha/\beta]_2 + [\gamma/\beta]_2$ (regard $[-]_2$ as $\mathbb{F}_2$-valued function, where $\mathbb{F}_2$ is the finite field with two elements). For $D = \pi_1 \cdots \pi_n$ and γ as above, put

$$S_\gamma^*(D) = \frac{I}{2}\sum_{c \in C}\frac{1}{\wp\left(\frac{c\omega}{D}\right) - I}\sum_T\left(\frac{c}{D_T}\right)_4^\gamma.$$

We could prove that

$$v_2(S_\gamma^*(D)) \geqslant ((3 + (-1)^\gamma)n - 2)/4.$$

Accordingly we define an $\mathbb{F}_2$-valued function $\varepsilon_n^{(\gamma)}$ as

$$\varepsilon_n^{(\gamma)}(D) = \begin{cases} (1 + (-1)^{\gamma-1})/2, & \text{if } v_2(S_\gamma^*(D)) = ((3 + (-1)^\gamma)n - 2)/4; \\ 0, & \text{if } v_2(S_\gamma^*(D)) > ((3 + (-1)^\gamma)n - 2)/4. \end{cases}$$

($n = n(D)$ is the number of distinct Gaussian prime factors of D). Then over Gaussian prime integers π, π_k congruent to 1 modulo 4 ($1 \leqslant k \in \mathbb{Z}$) (and their products), we define $\mathbb{F}_2$-valued functions s_1 and $\delta_n^{(\gamma)}$ ($n = 1, 2, \cdots$) inductively as follows

$$s_1(\pi) = \begin{cases} 1, & \text{if } v_2(\pi - 1) = 2, \\ 0, & \text{if } v_2(\pi - 1) > 2; \end{cases}$$

$$\delta_1^{(\gamma)}(\pi) = s_1(\pi) + \varepsilon_1^{(\gamma)}(\pi),$$

$$\delta_n^{(\gamma)}(D) = \delta_n^{(\gamma)}(\pi_1, \cdots, \pi_n) = \varepsilon_n^{(\gamma)}(D) + \sum_{\varnothing \neq T \subsetneqq \{1, \cdots, n\}} \Big(\prod_{\pi \mid \hat{D}_T} \Big[\frac{D_T}{\pi} \Big]_2 \Big) \delta_t^{(\gamma)}(D_T), \quad (n \geqslant 2),$$

where T runs over all non-trivial subsets of $\{1, \cdots, n\}$, $t = t(T) = \# T$ is the cardinal of T.

Theorem 2. For $D = \pi_1 \cdots \pi_n \in \mathbb{Z}[\sqrt{-1}]$ and rational integer $\gamma \not\equiv 0 \pmod 4$ as above, Let ψ_{D^γ} be the Hecke character of $\mathbb{Q}(\sqrt{-1})$ attached to the elliptic curve E_{D^γ}: $y^2 = x^3 - D^\gamma x$. Then for the 2-adic valuation of $L(\bar{\psi}_{D^\gamma}, 1)/\omega$ we have

(i) $v_2(L(\bar{\psi}_{D^\gamma}, 1)/\omega) \geqslant (3 + (-1)^\gamma)(n-1)/4$;

(ii) the equality in (i) holds if and only if $\delta_n^{(\gamma)}(D) = 1$.

Proposition 1. Let $D = \pm p_1 \cdots p_m \equiv 1 \pmod 4$ with $p_k \not\equiv 5 \pmod 8$ distinct positive rational prime numbers ($k = 1, \cdots, m$), $\gamma \in \mathbb{Z}$ and $\gamma \not\equiv 0 \pmod 4$. If $\delta_n^{(\gamma)}(D) = 1$, then the first part of the B-SD conjecture is true for the elliptic curve E_{D^γ}: $y^2 = x^3 - D^\gamma x$, that is

$$\operatorname{rank}(E_{D^\gamma}(\mathbb{Q})) = \operatorname{ord}_{s=1}(L(E_{D^\gamma}/\mathbb{Q}, s)) = 0.$$

(where $n = n(D)$ is the number of distinct Gaussian prime factors of D.)

2 L-series attached to curves $y^2 = x^3 - D^2 x$ with more general D

Consider the elliptic curves

$$E_{D^2}: y^2 = x^3 - D^2 x \quad \text{with} \quad D = \pi_1 \cdots \pi_n,$$

where $\pi_k \equiv 1 \pmod{2 + 2I}$ are distinct Gaussian (primitive) prime integers in $\mathbb{Z}[\sqrt{-1}]$ ($k = 1, \cdots, n$). Let $S = \{\pi_1, \cdots, \pi_n\}$. For any subset T of $\{1, \cdots, n\}$, define D_T and $\hat{D}_T$ as in Section 1 above. Let $L_S(\bar{\psi}_{D_T^2}, s)$ be the Hecke L-series of $\bar{\psi}_{D_T^2}$ (the complex conjugate of $\psi_{D_T^2}$) with the Euler factors omitted at all primes in S, where $\psi_{D_T^2}$ is the Hecke character of the Gaussian field $\mathbb{Q}(\sqrt{-1})$ attached to the elliptic curve $E_{D_T^2}$: $y^2 = x^3 - D_T^2 x$. Over Gaussian (primitive) primes π, π_k congruent to 1 modulo $\theta(= 2 + 2I)$ ($1 \leqslant k \in \mathbb{Z}$) (and their products), we define the $\mathbb{F}_2$-valued functions δ_n ($n = 1, 2, \cdots$) inductively as follows

$$\delta_1(\pi) = \begin{cases} 1, & \text{if } v_2(\pi - 1) = \frac{3}{2}; \\ 0, & \text{if } v_2(\pi - 1) > \frac{3}{2}. \end{cases}$$

$$\delta_n(D) = \delta_n(\pi_1, \cdots, \pi_n) = \sum_{\varnothing \neq T \subsetneqq \{1, \cdots, n\}} \Big(\prod_{\pi \mid \hat{D}_T} \Big[\frac{D_T}{\pi} \Big]_2 \Big) \delta_t(D_T), \quad (n \geqslant 2),$$

where T runs over all non-trivial subsets of $\{1, \cdots, n\}$, $t = t(T) = \# T$ is the cardinal of T.

Theorem 3. Let ψ_{D^2} be the Hecke character of $\mathbb{Q}(\sqrt{-1})$ attached to the elliptic curve E_{D^2}: $y^2 = x^3 - D^2 x$, where $D = \pi_1 \cdots \pi_n \in \mathbb{Z}[\sqrt{-1}]$ with $\pi_k \equiv 1 \pmod{2 + 2I}$ are distinct Gaussian (primitive) prime integers ($k = 1, \cdots, n$). Then for the 2-adic valuation of the values of the L-series we have

(i) $v_2(L(\bar{\psi}_{D^2}, 1)/\omega) \geqslant n - \frac{3}{2}$;

(ii) the equality in (i) holds if and only if $\delta_n(D) = 1$.

Proposition 2. Let D be any rational odd integer. If $\delta_n(\sigma D) = 1$ for $\sigma = 1$ or -1, then the first part of the B-SD conjecture is true for the elliptic curve E_{D^2}: $y^2 = x^3 - D^2 x$, that is

$$\operatorname{rank}(E_{D^2}(\mathbb{Q})) = \operatorname{ord}_{s=1}(L(E_{D^2}/\mathbb{Q}, s)) = 0.$$

(where $n = n(D)$ is the number of distinct Gaussian prime factors of D.)

3 L-series attached to curves $y^2 = x^3 - 2^4 3^3 D^\lambda$

Now we turn to the family of elliptic curves $y^2 = x^3 - D_2$ over $K = \mathbb{Q}(\sqrt{-3})$ with complex multiplication by $\sqrt{-3}$. Let $\tau = (-1 + \sqrt{-3})/2$ be a primitive cubic root of unity, and let $O_K = \mathbb{Z}[\tau]$ be the ring of integers of K. We study the elliptic curves

$$E_{D^\lambda}: y^2 = x^3 - 2^4 3^3 D^\lambda \quad \text{with} \quad D = \pi_1 \cdots \pi_n,$$

where $\pi_k \equiv 1 \pmod 6$ are distinct primes of O_K ($k = 1, \cdots, n$), $\lambda \in \mathbb{Z}$ and $\lambda \equiv \pm 2 \pmod 6$ (See Remark 1. (iii) in the following). Let $S = \{\pi_1, \cdots, \pi_n\}$. For any subset T of the set $\{1, \cdots, n\}$, define $D_T = \prod_{k \in T} \pi_k$, $\hat{D}_T = D/D_T$, $D_{\varnothing} = 1$. Let $\psi_{D_T^\lambda}$ be the Hecke character of K attached to the elliptic curve $E_{D_T^\lambda}$: $y^2 = x^3 - 2^4 3^3 D_T^\lambda$, and $L_S(\bar{\psi}_{D_T^\lambda}, s)$ be the Hecke L-series of $\bar{\psi}_{D_T^\lambda}$ with the Euler factors omitted at all primes in S (here $\bar{\psi}_{D_T^\lambda}$ is the complex conjugate of $\psi_{D_T^\lambda}$). Then $L_S(\bar{\psi}_{D_T^\lambda}, 1)$ could be expressed by the Weierstrass $\wp$-functions.

Theorem 4. For any factor D_T of $D = \pi_1 \cdots \pi_n \in \mathbb{Z}[\tau]$ and any rational integer $\lambda \equiv \pm 2 \pmod 6$ as above, let $\psi_{D_T^\lambda}$ be the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve $E_{D_T^\lambda}$: $y^2 = x^3 - 2^4 3^3 D_T^\lambda$. Then we have

$$\frac{D}{\Omega}\left(\frac{3}{D_T}\right)_3^\lambda L_S(\bar{\psi}_{D_T^\lambda}, 1) = \frac{1}{2\sqrt{3}} \sum_{c \in C} \left(\frac{c}{D_T}\right)_3^{\lambda/2} \frac{1}{\wp\left(\frac{c\Omega}{D}\right) - 1} + \frac{1}{3\sqrt{3}} \sum_{c \in C} \left(\frac{c}{D_T}\right)_3^{\lambda/2},$$

where $(-)_3$ is the cubic residue symbol, C is any complete set of representatives of the relatively prime residue classes of O_K modulo D, and $\wp(z)$ the Weierstrass $\wp$-function satisfying $\wp'(z)^2 = 4\wp(z)^3 - 1$ with period lattice $L_\Omega = \Omega O_K$ (corresponding to the elliptic curve E_1: $y^2 = x^3 - 1/4$), and $\Omega = 3.059908\cdots$ is an absolute constant.

Theorem 5. For $D = \pi_1 \cdots \pi_n \in \mathbb{Z}[\tau]$ and rational integer $\lambda \equiv \pm 2 \pmod 6$ as above, let ψ_{D^λ} be the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve E_{D^λ}: $y^2 = x^3 - 2^4 3^3 D^\lambda$. Then for the 3-adic valuation of the values of the L-series we have

$$v_3(L(\bar{\psi}_{D^\lambda}, 1)/\Omega) \geqslant \frac{n}{2} - 1.$$

Theorem 6. Let $D = \pi_1 \cdots \pi_n$, where $\pi_k \equiv 1 \pmod{6\sqrt{-3}}$ are distinct prime elements of $\mathbb{Z}[\tau]$ $(k = 1, \cdots, n)$, $\lambda \in \mathbb{Z}$ and $\lambda \equiv \pm 2 \pmod 6$, and let ψ_{D^λ} be the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve E_{D^λ}: $y^2 = x^3 - 2^4 3^3 D^\lambda$. Then for the 3-adic valuation of the values of the L-series we have

$$v_3(L(\bar{\psi}_{D^\lambda}, 1)/\Omega) \geqslant (n-1)/2.$$

Remark 1. (i) For the Hecke characters ψ_{D^γ} in Theorems 2, 3 and ψ_{D^λ} in Theorem 5, 6 respectively, by the results of Refs. [10 ~ 12], we know that $L(\bar{\psi}_{D^\gamma}, 1)/\omega$ and $L(\bar{\psi}_{D^\lambda}, 1)/\Omega$ are all algebraic numbers.

(ii) When $\gamma = 2$, the case in Theorems 1 and 2 above turns out to be the one studied in Ref. [6], and then in this case, if $\delta_n^{(2)}(D) = 1$, the Birch and Swinnerton-Dyer conjecture is true for E_{D^2} in Proposition 1, which was proved by Zhao in Ref. [6]. When $\gamma = 1$ and $\lambda = 2$, the cases in Theorems 1, 2, 4 and 5 turn out to be the ones studied in Ref. [7] and footnotes on page 785.

(iii) For any rational integer γ in $\mathbb{Z}$, by a simple change of variables, the elliptic curve E_{D^γ}: $y^2 = x^3 - D^\gamma x$ are always $\mathbb{Q}(\sqrt{-1})$-isomorphic to $E_{D^{\gamma_0}}$: $y^2 = x^3 - D^{\gamma_0} x$ with $\gamma_0 = 0$, 1, 2, or 3. In particular, when $\gamma_0 = 0$, $E_{D^{\gamma_0}} = E_1$: $y^2 = x^3 - x$, which is the trivial case. Therefore, the results in Theorems 1 and 2 above are reduced to the three essential cases $\gamma = 1$, 2 or 3. The case of λ is similar, that is, the results in Theorem 4 ~ 6 above are reduced to the two essential cases $\lambda = 2$ or 4.

Acknowledgements The author would like to thank Professor Zhang Xianke and Professor Feng Keqin for their help and encouragement, the author would also like to thank Professor Zhao Chunlai for his wonderful idea in his paper[6], which arouses the author's interest in this work.

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